MATH 3235 Probability Theory 10/06/22

Uniform in $[A, B]$

$$f(x) = \frac{1}{B-A} \qquad A \le x \le B$$

Exponential par λ

$$f(x) = \lambda e^{-\lambda x} \qquad x \geq 0$$

Normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

par μ, σ^2

$\mu = 0 \quad \sigma = 1$ Normal standard

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy = \Phi(x)$$

Probability Integral.

$$I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$$

$$I^2 = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy =$$

$$= \int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}} dx\,dy =$$

$$= \int_0^{2\pi} \int_0^{\infty} \rho e^{-\frac{\rho^2}{2}} d\rho = 2\pi \int_0^{\infty} \rho e^{-\frac{\rho^2}{2}} d\rho$$

$$= 2\pi \left(-e^{-\frac{\rho^2}{2}} \right) \Big|_0^{\infty} = 2\pi$$

X continuous r.v.

$g : \mathbb{R} \to \mathbb{R}$

$Y = g(X)$

f_Y p.d.f. of Y?

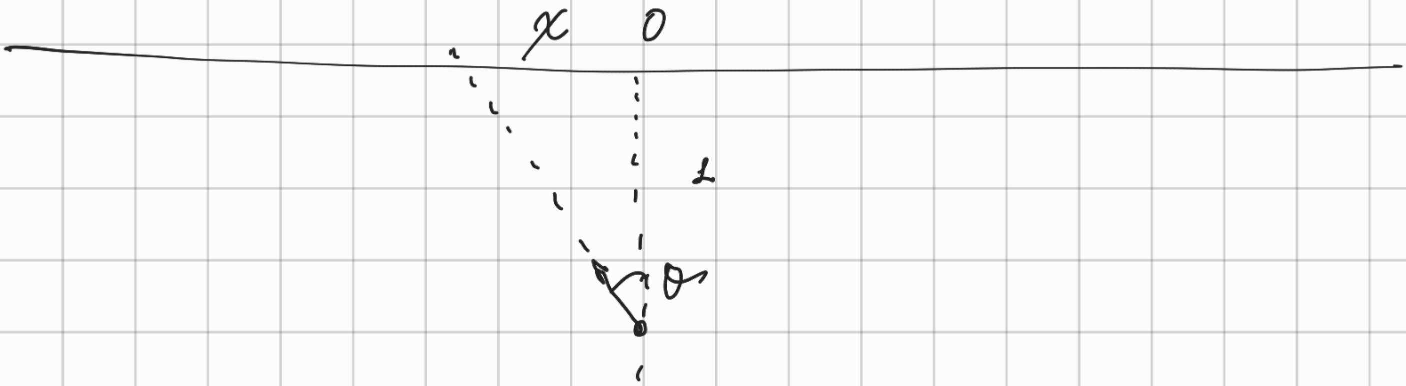
$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y) = \mathbb{P}(X \leq g^{-1}(y))$$
$$F_X(g^{-1}(y))$$

if g is invertible and increasing.

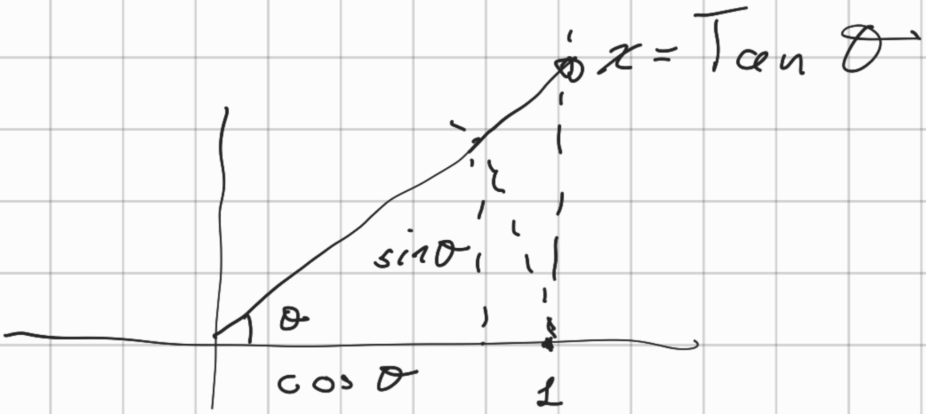
$g' \geq 0$ or $g' \leq 0$.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$$

0



Assuming That θ is uniform in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Distribution of X?



$$f_X(x) = \frac{1}{\pi} \frac{1}{1+x^2}$$

Cauchy distribution.

X r.v.

$Y = X^2$

$\mathbb{P}(Y \le y) =$

$\mathbb{P}(-\sqrt{y} \le X \le \sqrt{y}) =$

$F_X(\sqrt{y}) - F_X(-\sqrt{y})$

$$f_Y(y) = \frac{1}{2\sqrt{y}} \left(f_X(\sqrt{y}) - f_X(-\sqrt{y}) \right)$$

$Y = 2X$

$\mathbb{P}(Y \le y) = \mathbb{P}(X \le y/2) =$

$= F_X\left(\frac{y}{2}\right)$

$$f_Y(y) = \frac{1}{2} f_X\left(\frac{y}{2}\right)$$

Random number generator.

rnd() random48()

Uniformly distributed in $[0, 1]$.

I want a procedure to generate a random number with a different distribution f.

U uniform in $[0, 1]$

G such that $G(U) = X$ has density f given.

$$G = F^{-1}$$

$$X = F^{-1}(U)$$

$$\mathbb{P}(X \leq x) = \mathbb{P}(F^{-1}(U) \leq x) =$$

$$\mathbb{P}\left(U \geq x \right) = \mathbb{P}(U \leq 1-x)$$

Exponential r.v.

$$F(x) = 1 - e^{-\lambda x}$$

$$y = 1 - e^{-\lambda x} = x = \frac{1}{\lambda} \ln(1-y)$$

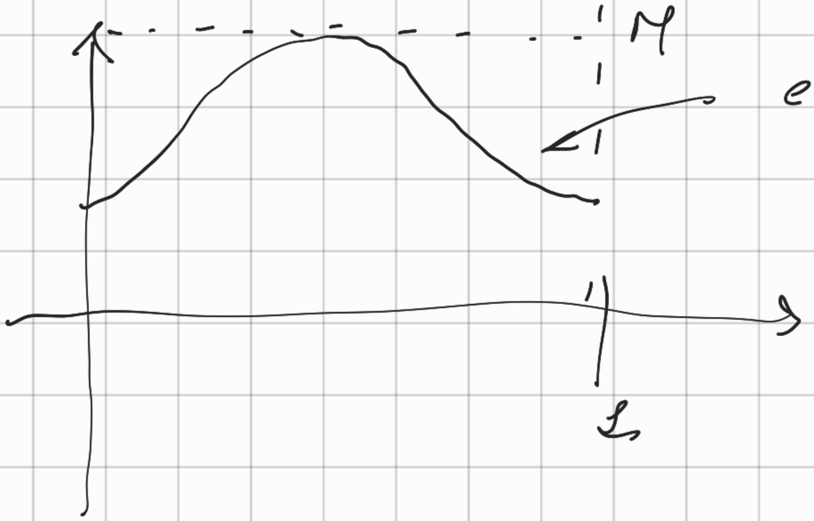
To generate an exponential r.v.

U is uniform

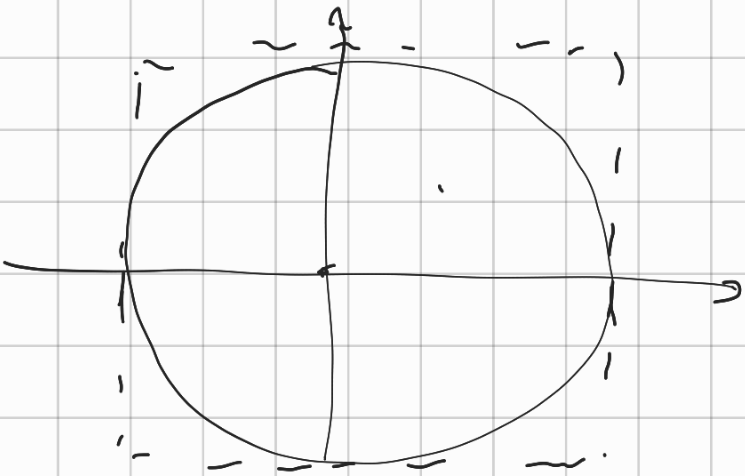
$X = \frac{1}{\lambda} \ln(U)$ is exponential.

Normal Standard X.

$$\Phi^{-1}$$



expression is heavy to compute



X Y uniform in [-1, 1].

I generate x_1 y_1

if $x_1^2 + y_1^2 < 1$ accept them

if $x_1^2 + y_1^2 > 1$ reject

generate x_2, y_2.

square has Area 4

circle " " π

in average you need $\frac{4}{\pi}$ iteration to have a result.

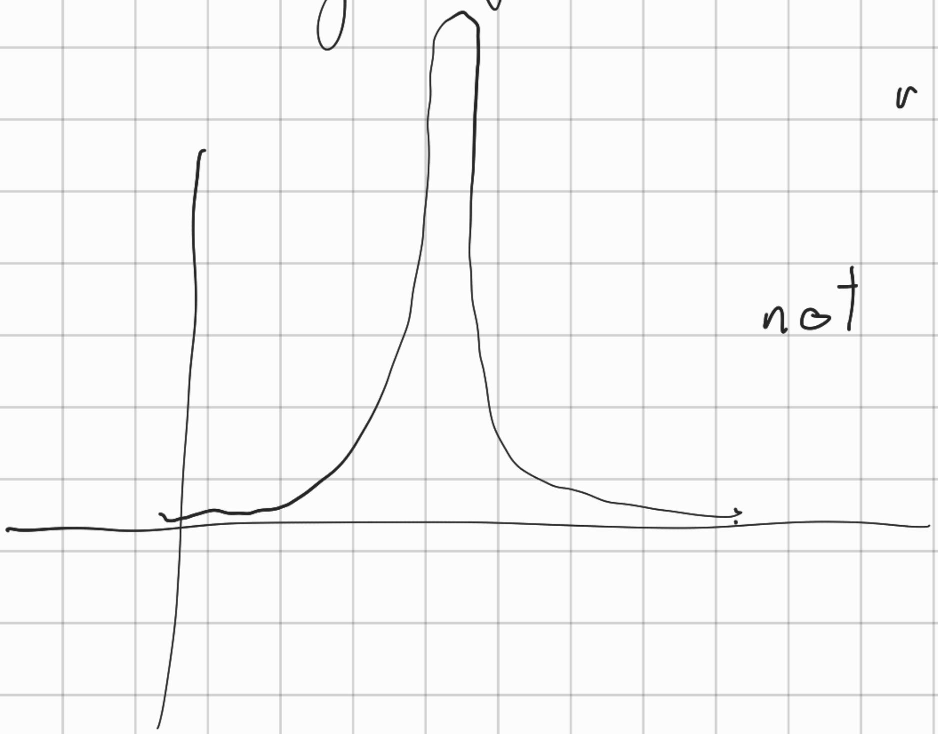
I want to generate f in $[0,1]$. I know $\sup f \leq M$

x y — x uniform in $[0,1]$

y uniform in $[0, M]$

if $y \leq f(x)$ accept

$y > f(x)$ reject and repeat.



not good!!

Expectations.

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

Thm

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

$$m_k = \mathbb{E}(X^k) = \int_{-\infty}^{\infty} x^k f(x)\, dx$$

$$\mathrm{var}(X) = m_2(X) - m_1(X)^2$$
$$= \mathbb{E}((X - \mathbb{E}(X))^2)$$

Suppose I have a Cauchy r.v. X

$$\mathbb{E}(X) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{1 + x^2}\, dx$$

$$\int_{-M}^{N} \frac{x}{1+x^2}\,dx = \frac{1}{2}\left(\ln\left(1+N^2\right) - \ln\left(1+M^2\right)\right)$$

The results depends on how you send M and N to infinity.

If X is Cauchy $\mathbb{E}(X)$ does not exists

$$\mathbb{E}(X^2) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x^2}{1+x^2}\,dx = +\infty$$